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On the Construction of Tables by the Method of Differences. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

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INTRODUCTION.

EVERY mathematical table consists of a series of values of a function corresponding to successive values of the variable, which last series of values forms the *argument* of the table. Any such table may be constructed therefore, when the function to be tabulated—which may be called the *characteristic function*—is known, by the evaluation of that function in terms of the successive values of the argument. It is only, however, when the table to be constructed is of limited extent that this method of formation would be employed. If the table be extensive, and especially if the characteristic function be complex, this, which may be called the *direct* method, would become too laborious, each value when formed in this way also requiring separate verification. In these circumstances the Method of Differences becomes available for the end in view. This method dispenses with all reference to the characteristic function beyond what is necessary for the formation of a few values (which I call *fundamental* or *primitive values*), at stated intervals; and in applying it, each value being dependent on the preceding, verification is obtained by the periodical coinci-

dence with those fundamental values of the corresponding terms in the series in course of being formed.

It is this method which I propose, in the following pages, to investigate and exemplify. It is an elementary application of the Calculus of Finite Differences, in the systematic treatises on which Calculus, however, it receives almost no attention. Mr. Woolhouse has contributed to the pages of this *Journal** a series of papers on "Interpolation, Summation, and the Adjustment of Numerical Tables," which are characterized by that gentleman's usual elegance and ability; but I think it will be found that his object in the papers referred to and that which I have here in view in no respect interfere with each other.

For the sake of perspicuity the matter of the following essay will be divided into four sections. In the first I shall explain and exemplify Horner's process for the transformation of functions. This process, which is one of great utility, is not nearly so well known as it deserves to be; and I hope the space devoted to its elucidation will not be considered wasted. In the second section the theorems of the Calculus of Finite Differences for which we have occasion will be briefly established. The third section will be devoted to the construction of tables in which the characteristic function is rational and integral. And the fourth section, to which the third will be only introductory, will be occupied with the construction of tables in which the characteristic function is irrational or transcendental. The two sections last named, especially the fourth, will comprise copious examples of the application of the principles established in the previous sections to the actual construction of tables.

SECTION I.—*On Horner's Method of Transformation.*

Problem I.

(1.) To divide, synthetically, $\phi(x)$, a given rational and integer polynomial of n dimensions in x, by the binomial $x-a$, where a is any number, positive or negative.

If the polynomial to be divided be

$$Ax^n+Bx^{n-1}+\ldots+Lx^2+Mx+N,$$

we shall have for quotient,

$$A_1x^{n-1}+B_1x^{n-2}+\ldots+K_1x^2+L_1x+M_1,$$

a polynomial of $n-1$ dimensions, and in which the coefficients are as yet undetermined; and for remainder, R_1. Hence—

* Vol. xi., pp. 61, 301; and vol. xii., p. 136.

$$Ax^n + Bx^{n-1} + \ldots + Lx^2 + Mx + N =$$
$$(A_1x^{n-1} + B_1x^{n-2} + \ldots + K_1x^2 + L_1x + M_1)(x - a) + R_1.$$

Performing the multiplication here indicated, the second member of this equation becomes

$$A_1x^n + (B_1 - A_1a)x^{n-1} + (C_1 - B_1a)x^{n-2} + \ldots + (L_1 - K_1a)x^2 +$$
$$(M_1 - L_1a)x + R_1 - M_1a.$$

Hence, equating coefficients, the two members being identical,

$$\begin{array}{ll} & A_1 = A \\ B_1 - A_1a = B & \therefore B_1 = A_1a + B \\ C_1 - B_1a = C & C_1 = B_1a + C \\ D_1 - C_1a = D & D_1 = C_1a + D \\ \vdots & \vdots \\ L_1 - K_1a = L & L_1 = K_1a + L \\ M_1 - L_1a = M & M_1 = L_1a + M \\ R_1 - M_1a = N & R_1 = M_1a + N \end{array}$$

(2.) It thus appears that the first coefficient of the quotient is equal to the first of the dividend; and that the others, including the remainder, are derived, each from that which precedes it, by a simple and uniform operation. This operation consists in multiplying each coefficient as it arises by a, and adding the product to the next coefficient of the dividend.

(3.) The process may be typified as follows :—

$$\begin{array}{lllllll} A & +B & +C \ldots & +L & +M & +N\,(a \\ & +Aa & +B_1a & +K_1a & +L_1a & +M_1a \\ \hline A & +B_1 & +C_1 \ldots & +L_1 & +M_1 & +R_1 \end{array}$$

where the coefficients A, B_1, C_1, &c., are respectively the sums of the quantities standing over them.

(4.) The following is a numerical example :—

Divide $2x^3 - 25x^2 - 4x + 50$ by $x - 10$.

$$\begin{array}{rrrl} 2 & -25 & -4 & +50\,(10 = a \\ & 20 & -50 & -540 \\ \hline 2 & -5 & -54 & -490 \end{array}$$

The quotient is $2x^2 - 5x - 54$, and the remainder is -490.

(5.) If a be essentially negative, by using $-a$ as the multiplier we obtain the quotient arising from division by $x + a$. Thus :—

$$\begin{array}{rrrl} 2 & -25 & -4 & +50\,(-10 \\ & -20 & +450 & -4460 \\ \hline 2 & -45 & +446 & -4410 \end{array}$$

The quotient here is $2x^2-45x+446$, and the remainder is -4410.

(6.) When the coefficients are not large numbers, and especially when the multiplier does not exceed 10, the foregoing operation may be very conveniently conducted without setting down the addends. It is recommended to practise this method, in accordance with which the example of (4) will assume the following form :—

$$\begin{array}{rrrl} 2 & -25 & -4 & +50\,(10 \\ & -5 & -54 & -490 \end{array}$$

(7.) In applying the foregoing process, in which the coefficients are detached from their arguments, to an *incomplete* polynomial—that is, a polynomial in which some of the powers of x are wanting—it is obvious that, to avoid error, it must be rendered *complete*, by indicating the places of the missing powers. This is done by inserting those powers, with zero coefficients. Thus, for the present purpose, $3x^4-2x^2+6x$ would require to be written $3x^4+0x^3-2x^2+6x+0$; and to divide this polynomial by $x-4$, we should proceed as follows :—

$$\begin{array}{rrrrl} 3 & 0 & -2 & 6 & 0\,(4 \\ & 12 & 46 & 190 & 760 \end{array}$$

The result is, quotient $3x^3+12x^2+46x+190$, and remainder 760.

(8.) Denoting the polynomial to be divided by $\phi(x)$, the quotient by Q, and the remainder by R, we obviously have

$$\phi(x)=Q(x-a)+R.$$

And if we write here a for x, we get

$$\phi(a)=R.$$

That is, the remainder arising from the division of $\phi(x)$ by $x-a$, is what $\phi(x)$ becomes by the substitution in it of a for x. Thus, in the examples of (4) and (5), -490 and -4410 are what the polynomial operated upon becomes when 10 and -10 are respectively substituted in it for x.

(9.) From the foregoing it appears, that besides an easy method of performing division by $x-a$, we have here also a compact and commodious process for substituting in any polynomial a specific value for the variable.

(10.) The theorem just proved is one of some interest and importance, and we shall have to refer to it hereafter. I therefore give another demonstration of it. Let the polynomial be

$$Ax^3+Bx^2+Cx+D.$$

Then, operating upon it as directed, the coefficients of the quotient will be—

$$\begin{array}{ll} \text{the first,} & A; \\ \text{second,} & Aa+B; \\ \text{third,} & (Aa+B)a+C=Aa^2+Ba+C; \end{array}$$

and the remainder will be

$$(Aa^2+Ba+C)a+D=Aa^3+Ba^2+Ca+D,$$

which is the given polynomial, with a substituted in it for x.*

Problem II.

(11.) To transform $\phi(x)$, a rational and integer function of x, of n dimensions, into $\phi(x+a)$; in other words, having given $\phi(x)=Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots+Mx+N$, to expand $\phi(x+a)$ in powers of x.

Divide $\phi(x)$ by $x-a$, and let the quotient and the remainder be Q_1 and R_1, respectively; divide Q_1 by $x-a$, and let the quotient and the remainder of this division be Q_2 and R_2 respectively. Proceed in this way, dividing each quotient as it arises by $x-a$, till Q_n and R_n shall have been formed. The required expansion will then be

$$\phi(x+a)=Ax^n+R_nx^{n-1}+R_{n-1}x^{n-2}+\ldots.+R_2x+R_1.$$

The rule may be exemplified thus:—

$$\begin{array}{rcc} x-a) & \phi(x) & \\ \hline x-a) & Q_1 & R_1 \\ \hline x-a) & Q_2 & R_2 \\ & \vdots & \vdots \\ \hline & Q_n & R_n \end{array}$$

(12.) From the foregoing operation we have

$$\begin{aligned} \phi(x)&=Q_1(x-a)+R_1; \\ Q_1&=Q_2(x-a)+R_2; \\ &\vdots \\ Q_{n-1}&=Q_n(x-a)+R_n. \end{aligned}$$

Hence, by successive substitution,

$$\begin{aligned} \phi(x)=Q_1(x-a)R_1&=Q_2(x-a)^2+R_2(x-a)+R_1 \\ &=Q_3(x-a)^3+R_3(x-a)^2+R_2(x-a)+R_1 \\ &\quad\vdots \\ &=Q_n(x-a)^n+R_n(x-a)^{n-1}+\ldots.+R_2(x-a)+R_1. \end{aligned}$$

* The process of this problem receives the name of Synthetic Division; and it is easily extended to the case in which the divisor is a polynomial of the form $x^m+ax^{m-1}+bx^{m-2}\ldots$ See Hutton's *Course*, by Davies, vol. ii., pp. 127 to 130, and 524; also *Penny Cyclopædia*, Supplement, article "Power."

Writing now in this equation, on both sides, $x+a$ for x, we obtain

$$\phi(x+a)=Q_n x^n+R_n x^{n-1}+R_{n-1}x^{n-2}+\ldots.+R_2 x+R_1.$$

Now, $Q_n=A$. For, owing to the form of the divisor, A, the leading coefficient of $\phi(x)$, is also the leading coefficient of each of the quotients. And Q_n being obviously of no dimensions with respect to x, it is a constant, and therefore equal to A. The expansion consequently is, finally,

$$\phi(x+a)=Ax^n+R_n x^{n-1}+\ldots.+R_2 x+R_1.$$

(13.) In applying this theorem to the transformation of functions, the divisions will be effected by the method of Problem I., in consequence of which the operation assumes a very commodious and compact form. The following is a type of it, in which, as before, the quantities beneath the lines are respectively the algebraical sums of the two above them.

Let $\phi(x)$ be of the fourth degree,

$$Ax^4+Bx^3+Cx^2+Dx+E.$$

$$\begin{array}{lllll}
{}^*A & B & C & D & E\ (a \\
 & Aa & B_1a & C_1a & D_1a \\
\hline
A & B_1 & C_1 & D_1 & R_1 \\
 & Aa & B_2a & C_2a & \\
\hline
A & B_2 & C_2 & R_2 & \\
 & Aa & B_3a & & \\
\hline
A & B_3 & R_3 & & \\
 & Aa & & & \\
\hline
A & R_4 & & &
\end{array}$$

Here

$$Q_1=Ax^3+B_1x^2+C_1x+D_1,$$
$$Q_2=Ax^2+B_2x+C_2,$$
$$Q_3=Ax+B_3,$$
$$Q_4=A;$$

and, the remainders being as indicated, the expansion is

$$\phi(x+a)=Ax^4+R_4x^3+R_3x^2+R_2x+R_1.$$

(14.) Comparing this with the expansion of the same function by Taylor's theorem, viz.,

$$\phi(x+a)=\phi(a)+\phi_1(a)x+\frac{\phi_2(a)}{1.2}x^2+\frac{\phi_3(a)}{1.2.3}x^3+\frac{\phi_4(a)}{1.2.3.4}x^4,$$

where $\phi_n(a)$ is the nth differential coefficient of $\phi(x)$ when a is put in it for x, we see that

* The first coefficient remaining constant throughout the operation, it is unnecessary in practice to repeat it on the successive lines.

$$R_1=\phi(a),\ R_2=\phi_1(a),\ R_3=\frac{\phi_2(a)}{1.2},\ R_4=\frac{\phi_3(a)}{1.2.3},\ \text{and } A=\frac{\phi_4(a)}{1.2.3.4}.$$

For none of these, however, except the first, shall we have any occasion; and it has been established already (8, 10).

(15.) For a numerical example, transform $\phi(x)=2x^3-25x^2-4x+50$, into $\phi(x+10)$.

2	−25	−4	+50 (10
	−5	−54	−490
	15	96	
	35		

$$\therefore\ \phi(x+10)=2x^3+35x^2+96x-490.$$

(16.) Again, transform $\phi(x)$ into $\phi(x-10)$.

2	−25	−4	+50 (−10
	−45	446	−4410
	−65	1096	
	−85		

$$\therefore\ \phi(x-10)=2x^3-85x^2+1096x-4410.$$

(17.) Looking to what is accomplished by the operation now exemplified, it is clear that we ought to have the same result whether we transform with a or with any numbers whose sum is a. For example, to transform $\phi(x)$ into $\phi(x+2)$, $\phi(x+2)$ into $\phi(\overline{x+2}+3)$ or $\phi(x+5)$, and $\phi(x+5)$ into $\phi(\overline{x+5}+5)$ or $\phi(x+10)$, ought to be the same thing as to transform, by a single operation, $\phi(x)$ into $\phi(x+10)$. That this is true the following shows:—

$\phi(x)$	2	−25	−4	+50 (2
		−21	−46	−42
		−17	−80	
		−13		
$\phi(x+2)$	2	−13	−80	−42 (3
		−7	−101	−345
		−1	−104	
		5		
$\phi(x+5)$	2	5	−104	−345 (5
		15	−29	−490
		25	96	
		35		
$\phi(x+10)$	2	35	96	−490, as before (15).

(18.) It will add to the interest connected with the process now elucidated if I diverge for a little from the object more immediately in view, to call attention to a very remarkable application of the process in question.

(19.) It has been already more than once shown (8, 10, 14), that the last term—the *absolute* term—of $\phi(x+a)$ is $\phi(a)$, the result of the substitution of a for x in $\phi(x)$. The same thing appears also if we make x in $\phi(x+a)$ equal to 0; for we then have simply $\phi(a)$. From this it follows, that if a be a root* of $\phi(x)$, the absolute term of $\phi(x+a)$ will be equal to nothing. Hence also, conversely, if the absolute term of $\phi(x+a)$ be nothing, it will follow that a, the number, or the sum of the numbers, used in the transformation, is a root of $\phi(x)$.

(20.) An instance will illustrate this. Let, as before,

$$\phi(x)=2x^3-25x^2-4x+50;$$

and transform it into $\phi(x+12\cdot5)$, thus—

	2	−25	−4	50 (12·5
		0	−4	0
		25	308·5	
		50		
$\phi(x+12\cdot5)$	2	50	308·5	0

or thus—

	2	−25	−4	50 (10	$50=\phi(0)$
		−5	−54	−490	$-490=\phi(10)$
		15	96		
		35			
$\phi(x+10)$	2	35	96	−490 (2	
		39	174	−142	$-142=\phi(12)$
		43	260		
		47			
$\phi(x+12)$	2	47	260	−142 (·5	
		48	284	0	$0=\phi(12\cdot5)$
		49	308·5		
		50			
$\phi(x+12\cdot5)$	2	50	308·5	0	

We thus find $\phi(x+12\cdot5)=2x^3+50x^2+308\cdot5x$, in which the absolute term is nothing; and 12·5 is a root of $\phi(x)$, as will be found on trial.

(21.) The transforming number a being necessarily rational, it will appear at once that we can fulfil the condition $\phi(a)=0$, or, in other words, cause the absolute term of $\phi(x+a)$ to vanish, only when $\phi(x)$ has a rational root. When the root or roots of $\phi(x)$

* By a root of a polynomial, as $\phi(x)$, is meant a number which, substituted in it for x, causes it to vanish. It is the same thing, therefore, as a root of the equation $\phi(x)=0$.

are irrational, however, these roots can always, as in other such cases, be represented by rational numbers, which shall fulfil the condition $\phi(x)=0$ as nearly as we please.

(22.) The result just established may be also deduced in a somewhat different manner. The substitution of any quantity r for x in $\phi(x+a)$ gives the same result as the substitution of $r+a$ for x in $\phi(x)$: each function then becomes $\phi(r+a)$. Hence, if r be a root of $\phi(x+a)$, $r+a$ will be a root of $\phi(x)$. If now r, still a root of $\phi(x+a)$, be equal to nothing, $0+a$, or a, will be a root of $\phi(x)$. But nothing cannot be a root of $\phi(x+a)$ unless the absolute term of this function be equal to nothing, as it is only terms having x as a factor that will vanish for $x=0$. We are hence conducted to the same conclusion as before, namely, that when $\phi(x+a)$ has nothing for its absolute term, a is a root of $\phi(x)$.

(23.) From what precedes, then, it appears that if we can, by a series of transformations upon any given function, $\phi(x)$, succeed in producing a transformed function, $\phi(x+a)$, whose absolute term is nothing, we shall have determined a root of $\phi(x)$; that is, we shall have obtained a solution of the equation $\phi(x)=0$, the root being a, the number, or the sum of the numbers, used in the transformation. And we shall have an additional root of $\phi(x)$ for every different value of a, which will enable us to bring $\phi(x+a)$ into the required form.

(24.) We should thus seem to be put in possession of a general method for the solution of numerical equations of all orders; and, with a little extraneous aid, such is the case. Before we can apply the transforming process with success, we require to know the *nature* and the *situation* of the roots of the equation to be solved; and this information we procure by the aid of considerations supplied in the theory of equations.

(25.) I now give an example of the solution of a cubic.

Required the roots of $2x^3-25x^2-4x+50$; that is, it is required to solve the equation,

$$2x^3-25x^2-4x+50=0.$$

Analyzing this equation, we learn, first, that the three roots are all real; secondly, that two of them are positive and one negative; and, thirdly, that the positive roots are comprised in the intervals [1, 2] and [10, 20] respectively; and the negative root in the interval [−2, −1]. I shall first develop the positive root in the interval [1, 2].

$\phi(x+0)$	2	-25	-4	50 (1·	$50=\phi(0)$
		-23	-27	23	$23=\phi(1)$
		-21	-48		
		-19			
$\phi(x+1)$	2	-19	-48	23(·4	
		$-18{\cdot}2$	$-55{\cdot}28$	·888	$\cdot 888=\phi(1{\cdot}4)$
		$-17{\cdot}4$	$-62{\cdot}24$		
		$-16{\cdot}6$			
$\phi(x+1{\cdot}4)$	2	$-16{\cdot}6$	$-62{\cdot}24$	·888(·01	
		$-16{\cdot}58$	$-62{\cdot}4058$	·263942	$\cdot 263942=\phi(1{\cdot}41)$
		$-16{\cdot}56$	$-62{\cdot}5714$		
		$-16{\cdot}54$			
$\phi(x+1{\cdot}41)$	2	$-16{\cdot}54$	$-62{\cdot}5714$	·263942	

(26.) The first three transformations, conducted in accordance with previous examples, are shown above. The object aimed at is to reduce the absolute term, 50, as nearly to zero as we can. The first transformation, by 1, reduces it to 23, $=\phi(1)$; the second, by ·4, to ·888, $=\phi(1{\cdot}4)$; and the third, by ·01, to ·263942, $=\phi(1{\cdot}41)$. The first transforming number, 1, is known, being the less of the limits comprising the root; the next, ·4, is the quotient of 23, the last coefficient, by 48, the preceding (anticipating, as well as we can, the effect of the next transformation); the third, ·01 is obtained in the same way, but with greater certainty; and, if we were to discontinue our transformations here, the work has now become so convergent that we should obtain the next three figures of the root true by the simple division of ·2639 . . . by 62·57 . . .

(27.) But it would be tedious to proceed further as above. The working can be much abbreviated and simplified by the aid of the following proposition in the theory of equations:—

If the terms of a polynomial (made complete (7), if necessary) be multiplied in order by the same number of successive terms of the series . . . m^{-3}, m^{-2}, m^{-1}, 1, m, m^2, m^3 . . . , the roots of the polynomial so transformed will be respectively those of the original polynomial multiplied by m.

(28.) Thus, the roots of x^3-7x+6 are 1, 2, and -3. And if, writing it x^3+0x^2-7x+6, we multiply the several terms by 10, 10^2, 10^3, 10^4, we get $10x^3-7000x+60000$, the roots of which will be found to be 10, 20, and -30.

(29.) It is usual, in employing this transformation, in order that the coefficients be not needlessly increased, to commence with unity as the first multiplier; so that the given polynomial would

have been with more propriety transformed into $x^3-700x+6000$, the roots of which are still, 10, 20, and -30.*

(30.) I now resume the equation in (25). The following is the complete solution, as far as twelve places:—

2	-25	-4	$50(1{\cdot}414,21,356237$
	-23	-27	23000
	-21	-4800	888000
	-190	-5528	263942000
	-182	-622400	13391888
	-174	-624058	850502
	-1660	-62571400	223398
	-1658	-62637528	35266
	-1656	$-6270362\cancel{4}$	3911
	-16540	-6270693	148
	-16532	$-627102\cancel{3}$	23
	-16524	-627104	4
	$-1\cancel{6}5\cancel{1}6$	$-62710\cancel{6}$	0

$$x=1{\cdot}41421356237$$

(31.) The first modification here is, that the formality of re-writing the coefficients of the several transformed polynomials is dispensed with, so that they no longer range in line with each other. I have indicated their connexion, however, by rows of dots. But this is not necessary, as will be seen.

(32.) A second modification is, that, by aid of the theorem in (27), the roots of each transformed polynomial as it arises are multiplied by 10; the effect of which is that each root figure is brought in succession into the unit's place, so far at least as to admit of its being employed in the next transformation as if belonging to that place. The ciphers annexed to the several coefficients for the purpose of this modification suffice to point out the completion of each transformation.

(33.) The third modification is but an adaptation of the second. When the last coefficient has been sufficiently extended to afford as many root figures as we want, the orderly multiplication of the roots is thenceforward effected by cutting off one figure from the first preceding coefficient, two from the second preceding, and so on. At the first curtailment, as here, the first coefficient usually disappears, and the operation becomes that for the solution of an equation of the next lower order—a quadratic. At the second the second coefficient here *nearly* disappears, and the operation

* The demonstration of the above theorem, which is very simple, will be found in any treatise on the theory of equations. The most accessible of these is probably Professor J. R. Young's *Analysis and Solution of Cubic and Biquadratic Equations.* This work, although strictly elementary, goes really a long way into the subject.

ultimately merges into contracted division, which gives the last six of the root figures. I have marked by commas in the root the points where the operations change.

(34.) It is apparent now, in this case at least—and it is also true *generally**—that it suffices to know the first figure of a root to enable us to evolve that root, figure by figure, either exactly, if rational, or, if irrational, to any degree of approximation that may be desired. I will illustrate this point. The second figure, 4, is suggested by the mental division of 23000 by 4800; and 4 succeeds, since the absolute term is reduced to 888, two places lower in the scale, and does not change sign, while it is easy to see that 5 would be too great. The next figure, 1, is suggested, with almost certainty, by division of 888... by 622..., and it succeeds. In fact this division would give the two figures, 1, 4. The next trial division, 2639... by 625... would give the three figures, 4, 2, 1. And thus as we advance the work becomes more and more convergent till, the first and second columns having disappeared, and so lost all power of influencing the third, the number at the bottom of this, 627106, being used with that *then* at the bottom of the last, namely, 223398, as divisor and dividend, the quotient is the last six figures of the root, namely 356237.

(35.) It is much less easy to *follow* a written example in an operation of this kind than to work it for oneself. I therefore suggest the working of the example to any one who wishes to master the process, and who finds difficulty in following the description.

(36.) I have finally here to point out that the absolute term has, as the result of the operation, been diminished till it has no significant figure in the first nine decimal places.

(37.) I shall now develop the negative root which we found to subsist in the interval $[-2, -1]$, and whose first figure is therefore -1, in the unit's place.

(38.) It is somewhat inconvenient to use a negative transforming number, and, therefore, when a negative root is to be evolved, it is customary, as a preliminary, to convert it into a positive root. This is done by the aid of the following theorem.

(39.) If the signs of the alternate terms of a polynomial (rendered complete) be changed, the signs of all the roots of that polynomial will be changed.

* It is not true *universally*, however. In rare cases two or more roots may be situated so closely together that a somewhat perplexing preliminary analysis is requisite to effect their complete separation. But the separation can *always* be made.

(40.) Thus, the roots of x^3-7x+6 being 1, 2, and -3, we shall have for the polynomial whose roots are -1, -2, and 3,

$$\begin{array}{rl} \text{either} & x^3-7x-6, \\ \text{or} & -x^3+7x+6; \end{array}$$

as the change may be made in either the first, third, &c., or the second, fourth, &c., terms.

(41.) So also we change the signs of the roots of the polynomial $2x^3-25x^2-4x+50$, by writing it $2x^3+25x^2-4x-50$; and its negative root, now become positive, is developed as follows:—

2	25	−4	−50(1·414,21,356238
	27	23	−27000
	29	5200	−1112000
	310	6472	−331058000
	318	777600	−16808112
	326	780942	−1067497
	3340	78428600	−280396
	3342	78562472	−44265
	3344	7869637$\cancel{8}$	−4910
	33460	7870307	−187
	33468	787097$\cancel{7}$	−30
	33476	787101	−6
	3$\cancel{3}$4$\cancel{8}$4	78710$\cancel{4}$	0

(42.) The analogy of this development to that in (30) supersedes the necessity for any general remarks here. The identity, except as to sign, of the two roots will, of course, be observed. This would hardly arise except in a polynomial specially framed, as the one whose roots are being developed has been. It is the product $(2x-25)(x^2-2)$, or $2(x-12{\cdot}5)(x-\sqrt{2})(x+\sqrt{2})$, of which the three roots obviously are

$$12{\cdot}5, \ \sqrt{2}, \text{ and } -\sqrt{2}.$$

It is the last two roots, $\pm\sqrt{2}$, that have been developed in (30) and (41). The last figure is truly 7, as in (30). The roots could have easily been carried to double the number of decimal places.

(43.) The rational root, 12·5 was developed in (20). I repeat the development here, for the purpose of introducing a modified application of the theorem in (27). By multiplying the coefficients of a polynomial by successive terms of the series $\ldots m^{-2}, m^{-1}, 1, m, m^2 \ldots$ *in direct order*, we multiply the roots of the polynomial by m. But also, if we use the terms of the series *in inverse order*, we divide the roots by m.

(44.) So, the first figure of the root now to be developed being in the tens' place, we commence by dividing the roots by 10, that we

may use this first figure in the transformation as if it belonged to the unit's place.

(45.) The coefficients are

$$2 \qquad -25 \qquad -4 \qquad 50;$$

and using as multipliers 10^2, 10, 1, 10^{-1}, they are changed as desired, and the operation is as follows:—

200	−250	−4	5 (12·5
	−50	−54	−490
	150	96	142000
2~~00~~	35~~0~~	174	0
	39	26000	
	43	28400	
	470		

(46.) The first transformation (by 1) here gives

$$200 \qquad 350 \qquad 96 \qquad -49;$$

and we now *multiply* the roots by 10 by using 10^{-2}, 10^{-1}, 1, and 10, as multipliers, which gives

$$2 \qquad 35 \qquad 96 \qquad -490.$$

We then proceed as usual; and the absolute term disappearing, we know that 12·5 is a root (19, 22).

(47). It deserves to be noticed, that the effect of the first transformation is to increase the absolute term and introduce in it a change of sign: from 5 it has become —49. This is an indication that a root has been passed over, situated in the interval [0, 10], which is the root developed in (30).

(48.) Mr. Horner's paper, giving an account of his method of solving equations, was published in the *Philosophical Transactions* for 1819.* The attention of mathematicians was first specially called to the method by Professor J. R. Young, in his *Algebra*, published about 1823. The same gentleman has also, in several subsequent works—one of which is referred to in the Note appended to (29)—amply explained and illustrated it. The late Peter Nicholson published several tracts on the subject of it, putting forward at the same time a mistaken claim to the origination of it; and it was introduced by the late T. S. Davies into the second volume of his edition of *Hutton's Course*. A history of the Problem of Evolution from Vieta to Horner, which contains many interesting details, was contributed by Professor De Morgan to the *Companion to the Almanac* for 1839; and the subject is continued and amplified, and many illustrations given, in the article "Involution and Evolution,"

* It was republished in the Appendix to the *Ladies' Diary* for 1838. It must be admitted to be in an unattractive and needlessly transcendental form.

in the *Penny Cyclopædia* and the *Supplement* to the same work. These two have since been combined in the article under the same title in the *English Cyclopædia* ("Arts and Sciences").

(49). Till very recently, when, as I am informed, it has been introduced by Mr. Todhunter in his *Theory of Equations,* Mr. Horner's method has been entirely ignored by all the University writers. It is not for me to suggest a reason for this; but it is certainly, in the circumstances, not a little singular to find retained in the text books the antiquated and laborious methods of transformation and solution, instead of Horner's elegant and efficient processes.

On the Construction of Tables of Mortality. By W. S. B. Woolhouse, F.R.A.S., F.S.S., *Vice-President of the Institute of Actuaries, &c.*

[Read before the Institute, 30th April, 1866.]

A TABLE of mortality is designed to represent the number of lives which, according to the best deductions from past experience, may be expected to survive at the termination of each successive year of age, supposing these survivors to be derived exclusively from a certain number of persons originally taken either at birth or at a given age. As a practical index of the mathematical law of average mortality such a table may be taken as a trustworthy guide for the future, provided that the number of lives which enter into its formation be sufficiently large and the particulars respecting them be correctly registered; and, what is equally important, that the observations shall extend over a considerable number of years.

Whether theoretically or practically considered, a table of mortality presents, in the most simple, complete and convenient form, the elementary data requisite for all computations involving the contingencies of life, and it is, therefore, universally adopted as the primitive basis upon which the whole superstructure of life assurance is built. In fact, it is the general permanence in this progressive distribution of mortality, between narrow probable limits of divergence, that constitutes the main stability and efficiency of Life Assurance Institutions. It is consequently needless to insist on the importance of accumulating reliable facts as to the statistics of human life, whenever that can be done on a large and efficient, scale, with the view of obtaining more perfect tables, or of more strictly testing the general accuracy of those in ordinary use, upon which an enormous amount of monied transactions is necessarily made to depend.